Journal of Engineering Mathematics **46:** 253–268, 2003.

A Dirichlet problem for the biharmonic equation in a semi-infinite strip

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Received 3 October 2002; accepted in revised form 3 April 2003

Abstract. This paper addresses the two-dimensional biharmonic problem for a semi-infinite strip with Dirichlet boundary conditions. The method of superposition is used to solve the problem. The object of this paper is to clarify mathematical questions connected with the solution of a special integral equation and to provide a rigorous justification of the applicability of the method of superposition. Mellin's transform technique of investigating the asymptotic behaviour of unknown density when the argument tends to infinity is used.

Key words: biharmonic equation, method of superposition, semi-infinite strip.

1. Introduction

The problem of solving the biharmonic equation in a semi-infinite strip arises both in elasticity studies and in studies of slow viscous flow. In the present paper we consider the Dirichlet problem for the homogeneous biharmonic equation in the half-strip $S = \{(x, y) : x > 0, |y| < 1\}$:

$$\begin{aligned} &\Delta^2 u(x, y) = 0, \quad (x, y) \in S \\ &u(x, \pm 1) = u_y(x, \pm 1) = 0, \quad x > 0, \\ &u(0, y) = f(y), \quad u_x(0, y) = g(y), \quad |y| < 1, \quad u(\infty, y) = 0. \end{aligned} \tag{1}$$

The main result of the present paper consist, in the algorithm for the construction of the solution of that boundary-value problem. Our problem is closely related to a number of studies in elasticity involving semi-infinite strips, *e.g.* [1–11], using eigenfunction expansions. In order to solve the boundary-value problem (1) we use the superposition method. This method for solving the three-dimensional problem of the equilibrium of an elastic rectangular parallelepiped was proposed by Lamé [12].

The main idea of the superposition method for the biharmonic equation in a rectangular two-dimensional domain is that of using the sum of two Fourier representations involving trigonometric functions in x- and y-coordinates (see [13-15] for detailed reviews of such studies). As regards the superposition method for the biharmonic equation in the halfstrip, the solution $u(x, y)$ is represented by the sum of a Fourier series in the y-coordinate and a Fourier integral in the x-coordinate (for details see [11, 16–19]). This sum and integral satisfy identically the governing biharmonic equation and together have sufficient functional arbitrariness for fulfilling the boundary conditions at the edges. The determination of the series coefficients and the densities of the integrals, representing the solution of the boundary-value problem, is reduced to the solution of a system of integral-algebraic equations. In this paper we prove that this system possesses a unique solution and the asymptotic properties of the solution are investigated. As a result, the smoothness properties of the boundary data are given under which it is possible to construct the solution of the boundary-value problem by the method of superposition. In this respect the Dirichlet problem is considered under the weakest assumptions.

Section 2 formulates the representation of the solution and shows that this depends ultimately on an integral equation. It is shown that this equation has a unique solution. Section 3 studies properties of the solution form: it is proved therein that it, indeed, satisfies rigorously the biharmonic equation, and that it, indeed, satisfies the boundary conditions at the lateral edges of the strip and at infinity. The principal result of Section 4 concerns the satisfaction of the non-zero boundary conditions at the edge $x = 0$.

To simplify the formulae we consider the symmetric case, *i.e.*,

$$f(y) = f(-y), \quad g(y) = g(-y), \quad |y| < 1.$$

In the remainder of this paper we assume the complex-valued functions f and g to satisfy the conditions

$$f \in W_p^1, \quad g \in L_p = L_p[-1,1], \quad p \in (1,\infty), \quad f(1) = 0, \tag{2}$$

where $W_p^k = W_p^k[-1,1]$, $k = 1, 2, \ldots$, is the Sobolev space with norm

$$\|f\|_{W_p^k} = \left\{ \sum_{m=0}^{k} \int_{-1}^{1} |f^{(m)}(y)|^p \mathrm{d}y \right\}^{1/p}.$$

We assume the following condition to be fulfilled

$$\int_{-1}^{1} g(y)\mathrm{d}y = 0, \tag{3}$$

which does not reduce the generality of the problem (see Remarks 5). Using $p \in (1,\infty)$ we have

$$p' = p(p-1)^{-1}, \quad p_0 = \min\{p, 2\},$$
$$N_p^k(f,g) = \|f\|_{W_p^k} + \|g\|_{W_p^{k-1}}, \quad k = 1, 2, \ldots .$$

To construct the solution of boundary-value problem (1) by the method of superposition, we use the following Fourier-series expansion for even functions f and g:

$$f(y) = f_0 + \sum_{n=1}^{\infty} f_n \cos(\alpha_n y), \; g(y) = \sum_{n=1}^{\infty} g_n \cos(\alpha_n y), \; \alpha_n = \pi n, \; n = 1, 2, \ldots$$

Due to conditions (2) and the Hausdorff-Young inequality [20, Section 13.5] we have

$$\{\alpha_n f_n\}_{n=1}^{\infty} = \left\{f_n^{(1)}\right\}_{n=1}^{\infty} \in l_{p_0'} : \; \sum_{n=1}^{\infty} |f_n^{(1)}|^{p_0'} < \infty, \tag{5}$$

where $f_n^{(1)}$ stand for the Fourier coefficients of the odd function $f'(y) \in L_p = L_p(-1,1]$ expanded in the orthonormal system $\{\sin(\alpha_n y)\}_{n=1}^{\infty}$.

2. The method of superposition

According to the general scheme of the method of superposition, omitting the intermediary formulae analogous to [18, Chapter 8], we look for a solution of the boundary-value problem (1) in the form

$$u = u_1 + u_2, \tag{6}$$

$$u_1(x, y) = \frac{1}{2\pi}\int_0^\infty \frac{U(s, y)X(s)}{s^3 \sinh^2 s}\cos(xs)\mathrm{d}s, \tag{7}$$

$$u_2(x, y) = \sum_{n=1}^{\infty}\left\{\left(\frac{(-1)^{n+1}X_n}{2\alpha_n^3} + f_n\right)(1 + \alpha_n x) + g_n x\right\}\mathrm{e}^{-\alpha_n x}\cos(\alpha_n y), \tag{8}$$

where

$$U(s, y) = (s\ \cosh\ s + \sinh\ s)\cosh\ sy - sy\ \sinh\ s\ \sinh\ sy \tag{9}$$

and the unknowns $X(s)$ and X_n. If, for example, the conditions

$$\int_0^\infty s^{-4}|X(s)|\mathrm{d}s < \infty, \quad \sum_{n=1}^{\infty}\alpha_n^{-3}|X_n| < \infty, \tag{10}$$

are satisfied, then the (even with respect to y) function $u(x, y)$ is indefinitely differentiable and satisfies in S the homogeneous biharmonic equation and the condition $u \to 0$, $x \to \infty$, $\forall y \in [-1, 1]$, and (formally) boundary conditions from (1) for the normal derivative.

The boundary conditions for the values of u at the boundary of the half-strip S lead, due to

$$\frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}sx}\mathrm{d}s}{s^2 + \alpha^2} = \mathrm{e}^{-\alpha x}, \ \alpha > 0, \ x > 0, \tag{11}$$

$$\frac{U(s, y)}{\sinh^2 s} = \frac{2}{s} + 4s^3\sum_{n=1}^{\infty}\frac{(-1)^n\cos(\alpha_n y)}{(s^2 + \alpha_n^2)^2}, \ |y| \le 1, \tag{12}$$

to the system of linear integral-algebraic equations

$$X(s) = -\frac{4s^3\sinh^2 s}{\Delta(s)}\sum_{n=1}^{\infty}\frac{X_n}{(s^2 + \alpha_n^2)^2} = F(s), \ s > 0, \tag{13}$$

$$X_n = \frac{4\alpha_n^3}{\pi}\int_0^\infty\frac{X(s)\mathrm{d}s}{(s^2 + \alpha_n^2)^2}, \quad n = 1, 2, \dots , \tag{14}$$

$$\frac{1}{\pi}\int_0^\infty s^{-4}X(s)\mathrm{d}s = f_0, \tag{15}$$

where

$$\Delta(s) = \sinh\ s\ \cosh\ s + s, \tag{16}$$

$$\begin{aligned} F(s) &= \frac{\sinh^2 s}{\Delta(s)} F_1(s), \quad F_1(s) = 4s^3 T_0(s), \\ T_0(s) &= -\sum_{n=1}^{\infty} (-1)^n \frac{2\alpha_n^3 f_n + (\alpha_n^2 - s^2) g_n}{(s^2 + \alpha_n^2)^2}. \end{aligned} \tag{17}$$

Evidently, system (13), (14) can be reduced to the following integral equation for the unknown function $X(s)$:

$$X(s) - \int_0^{\infty} Q(s,t) X(t) \mathrm{d}t = F(s), \quad s > 0, \tag{18}$$

and the kernel

$$Q(s,t) = \frac{16 s^3 \sinh^2 s}{\pi \Delta(s)} \sum_{n=1}^{\infty} \frac{\alpha_n^3}{(s^2 + \alpha_n^2)^2 (t^2 + \alpha_n^2)^2}.$$

Concerning conditions (10), (15), it will be shown below (see Lemma 1) that they are fulfilled automatically, if $X(s)$ is the solution of Equation (18) which belongs to the Hilbert space $H_\sigma = L_2(R_+; t^{2\sigma-1})$, $\sigma \in (-2, -1, 1/p_0)$ with norm

$$\|X\|_{H_\sigma} = \|X(s) s^{\sigma - 1/}\|_{L_2(R_+)} = \left\{ \int_0^{\infty} |X(s)|^2 s^{2s\sigma - 1} \mathrm{d}s \right\}^{1/2}. \tag{19}$$

Let λ_k, $k = 1, 2, \ldots$, be the roots of the function $\Delta(\lambda)$ lying in the half-plane $\mathfrak{Im}\lambda > 0$. According to [21], the following relation is valid

$$\lambda_k = (-1)^k \pi k + \frac{\mathrm{i}}{2} \log k + \mathrm{O}(1), \quad k \to \infty. \tag{20}$$

Due to (20) there exists a number $\epsilon_0 > 0$, such that the function $\sinh^2 \lambda / \Delta(\lambda)$ is analytic and bounded in the sector

$$\Sigma_{\epsilon_0} = \{\lambda \in \mathrm{C} : |\mathfrak{Im}\, \lambda| < \epsilon_0 \mathfrak{Re}\, \lambda\},$$

and there exists $\delta > 0$ such that

$$|1 - \sinh^2 \lambda / \Delta(\lambda)| \le c \mathrm{e}^{-\delta|\lambda|}, \quad \lambda \in \Sigma_{\epsilon_0}. \tag{21}$$

We shall often use the following relations [22, pp. 298, 687]:

$$\int_0^{\infty} \frac{s^{\gamma+2} \mathrm{d}s}{(s^2+1)^2} = \frac{\pi(\gamma+1)}{4 \cos(\pi\gamma/2)}, \quad \mathfrak{Re}\, \gamma \in (-3, 1), \tag{22}$$

$$\sum_{n=1}^{\infty} \frac{1}{(\lambda^2 + \alpha_n^2)^2} = \frac{1}{2\lambda^3} \left(\frac{\sinh \lambda \cosh \lambda + \lambda}{2 \sinh^2 \lambda} - \frac{1}{\lambda} \right). \tag{23}$$

Lemma 1. Let $X(s) \in H_\sigma$, where $\sigma \in (-2, -1 - 1/p_0)$, be the solution of the integral equation (18) with the right-hand side from (17). Then $X(s)$ admits meromorphic continuation onto the entire complex plane with the poles at $\pm\lambda_k$. Moreover, conditions (10), (15) are valid as well as the estimate

$$|X(\lambda)| \le c \left\{ \|X\|_{H_\sigma} + N_p^1(f, g) \right\} \cdot |\lambda|^4 (1 + |\lambda|)^{-4-\sigma}, \quad \lambda \in \Sigma_{\epsilon_0}, \tag{24}$$

with the constant $c > 0$, independent of $X(s)$, $f(y)$ and $g(y)$.

Proof. Applying the Cauchy-Bounjakowsky inequality to the sequence X_n, defined via $X(s)$ by Equations (14), we obtain the following estimate

$$|X_n| \le c\|X(s)\|_{H_\sigma} \cdot n^{|\sigma|}, \quad n = 1, 2, \dots . \tag{25}$$

Then (17) and (18) imply that $X(s)$ admits the meromorphic continuation onto the complex plane according to

$$X(\lambda) = \frac{4\lambda^3 \sinh^2 \lambda}{\Delta(\lambda)} T(\lambda), \quad T(\lambda) = \sum_{n=1}^{\infty} \frac{X_n}{(\lambda^2 + \alpha_n^2)^2} + T_0(\lambda), \tag{26}$$

where the function $T_0(\lambda)$ is given by (17). It follows from (26) that the poles of $X(\lambda)$ are $\pm\lambda_k$. Then from (5) and (17) we obtain for each $\epsilon > 0$:

$$|T_0(\lambda)| \le c_\epsilon N_p^1(f, g)(1 + |\lambda|)^{-2+1/p_0}, \quad \lambda \in \Sigma_\epsilon. \tag{27}$$

On the other hand, from (25–27), we obtain (24). Condition (10) is a consequence of the estimates (24), (25).

To prove (15) let us divide the integral equation (18) by the function $2\pi^3 \sinh^2 s/\Delta(s)$ and integrate the resulting equation with respect to $s \in (0, \infty)$. Then, using the definition of the function $F_1(s)$ (see (17)) and Equations (22), (23), we obtain

$$\frac{1}{\pi} \int_0^\infty s^{-4} X(s) \mathrm{d}s = -\sum_{n=1}^{\infty} (-1)^n f_n = f_0.$$

The Lemma is thus proved.

Thus, the construction of the solution of the boundary-value problem (1) by the method of superposition requires one to investigate the properties of the integral equation (18) with the right-hand side from (17). Let us notice that the kernel of Equation (18) is nonnegative and satisfies the regularity condition (see 22), (23))

$$1 - \int_0^\infty Q(s, t) \mathrm{d}t = \frac{2 \sinh^2 s}{s\Delta(s)} > 0, \quad s \ge 0.$$

The kernel $Q(s, t)$ can be presented in the form

$$Q(s, t) = Q_0(s, t) + \left(\frac{\sinh^2 s}{\Delta(s)} - 1 \right) Q_0(s, t),$$

where the main part

$$Q_0(s, t) = \frac{16 s^3}{\pi} \sum_{n=1}^{\infty} \frac{\alpha_n^3}{(s^2 + \alpha_n^2)^2 (t^2 + \alpha_n^2)^2} = \frac{1}{t} \sum_{n=1}^{\infty} \frac{1}{n} \phi_1 \left(\frac{s}{n} \right) \phi_2 \left(\frac{n}{t} \right), \tag{28}$$

and the functions

$$\phi_1(\lambda) = \frac{4\pi\lambda^3}{(\lambda^2 + \pi^2)^2}, \quad \phi_2(\lambda) = \frac{4\pi\lambda^3}{(\pi^2\lambda^2 + 1)^2}, \quad \Re\mathfrak{e}\, \lambda > 0.$$

On the other hand, due to (22), we have for the function

$$D(\gamma) = 1 - \left(\int_0^\infty \phi_1(s) s^{\gamma-1} \mathrm{d}s \right) \cdot \left(\int_0^\infty \phi_2(s) s^{\gamma-1} \mathrm{d}s \right), \ \Re\mathfrak{e}\, \gamma \in (-3, 1),$$

the following equation

$$D(\gamma) = \frac{D_0(\gamma)}{\cos^2 \pi\gamma/2}, \ \ D_0(\gamma) = \cos^2 \pi\gamma/2 - (\gamma+1)^2. \tag{29}$$

In what follows it is important that, according to [23], there exists a $\sigma_0 \in (1, 2)$ such that

$$D(\gamma) \neq 0, \quad \gamma \neq 0, \quad \mathfrak{Re}\,\gamma \in (-2, \sigma_0). \tag{30}$$

Also, if $\gamma_k, k = 1, 2, \ldots$ are are the roots of the functions $D(\gamma)$ in the half-plane $\mathfrak{Re}\gamma > 0$, then $\mathfrak{Re}\,\gamma_k \to +\infty, k \to \infty$.

Since $D(1 + \mathrm{i}s)$ is a real function of the variable $s \in \mathrm{R}$, it follows from (30) that the index

$$\kappa_\sigma = -\frac{1}{2\pi}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{D'(\gamma)}{D(\gamma)}\mathrm{d}\gamma = 0, \ \ \forall\sigma \in (-2, 0).$$

Thus, according to [24], we obtain the following statement.

Theorem 1. For each $\sigma \in (-2, 0)$ Equation (18) has a unique solution $X \in H_\sigma$ for any $F \in H_\sigma$.

Corollary 1. Let $\sigma \in (-2, -1 - 1/p_0)$. Then for any f, g, Equation (18), with the right-hand side as in (17), possesses a unique solution $X(s) \in H_\sigma$.

Remark 1. If $X_j(s) \in H_\sigma$, $j = 1, 2$ are the solutions of Equation (18) with the same right-hand part (17) and $\sigma_j \in (-2, -1-1/p_0)$, then it follows from Theorem 1 and from (24) that $X_1(s) = X_2(s)$.

3. Properties of the solution of the integral equation

In what follows by the solution of Equation (18) we understand a function $X(s) \in H_\sigma$, for some fixed $\sigma \in (-2, -1 - 1/p_0)$, which is the solution of integral equation (18) with the right-hand part (17) (see Corollary 1).

According to [24], in order to investigate the properties of the solution $X(s)$ one should consider the analytic properties of the Mellin transform of the function $F(s)$ defined by (16), (17). The existence of the Mellin transform

$$G_1(\gamma) = \int_0^\infty F_1(s)s^{\gamma-1}\mathrm{d}, \quad \mathfrak{Re}\,\gamma \in (-3, -1 - 1/p_0),$$

follows from estimate 27). In what follows we need the following Proposition (see also [25, Section 1.11]).

Proposition 1. For each $y \in (-1, 1)$ the functions

$$S(\gamma, y) = \sum_{n=1}^{\infty}(-1)^n\alpha_n^\gamma \sin(\alpha_n y), \ \ C(\gamma, y) = \sum_{n=1}^{\infty}(-1)^n\alpha_n^\gamma \cos(\alpha_n y),$$

with $\mathfrak{Re}\,\gamma < -1$ admit analytic continuation onto the whole complex γ-plane and for an $\mu \geq 0$ and $\delta > 0$ the following estimates are valid:

$$|S(\gamma, y)| + |C(\gamma, y)| \leq c_{\mu,\delta} \cdot \mathrm{e}^{\delta|\mathfrak{Im}\gamma|}(1 - |y|)^{-(1+\mu)}, \quad \mathfrak{Re}\,\gamma \leq \mu.$$

Moreover, for $\mathfrak{Re}\,\gamma > -1$, $|y| < 1$, the following are true:

$$S(\gamma, y) = -\Gamma(\gamma+1)\cos(\pi\gamma/2)\cdot\frac{(1+y)^{\gamma+1}-(1-y)^{\gamma+1}}{\pi(1-y^2)^{\gamma+1}} + S_1(\gamma, y),$$

$$C(\gamma, y) - C(\gamma, 0) = -\Gamma(\gamma+1)\sin(\pi\gamma/2)\cdot\frac{(1+\gamma)^{\gamma+1}+(1-y)^{\gamma+1}}{\pi(1-y^2)^{\gamma+1}} + C_1(\gamma, y), \qquad (31)$$

where for any $\mu > 0$ and $\delta > 0$:

$$|S_1(\gamma, y)| + |C_1(\gamma, y)| \le c_{\mu,\delta}\cdot e^{\delta|\mathfrak{Im}\,\gamma|}, \quad \mathfrak{Re}\,\gamma \in (-1+1/\mu, \mu).$$

The above proposition is used to the establishment of the following Lemma.

Lemma 2. The function $G_1(\gamma)$ is analytic in the half-strip $\mathfrak{Re}\,\gamma \in (-3, -1-1/p)$ and

$$|G_1(\gamma)| \le c e^{-(\pi/2-\delta)|\mathfrak{Im}\,\gamma|}, \ \mathfrak{Re}\,\gamma \in [\sigma_1, \sigma_1] \qquad (32)$$

for any $\delta \in (0, \pi/2)$ and $[\sigma_1, \sigma_1] \subset (-3, -1-1/p)$. For $\mathfrak{Re}\,\gamma > -2$ the representation

$$G_1(\gamma) = 2\Gamma(\gamma+2)\left\{(\gamma+1)\tan\frac{\pi\gamma}{2}\int_{-1}^{1} f'(y)\frac{(1+y)^{\gamma+1}-(1-y)^{\gamma+1}}{\pi(1-y^2)^{\gamma+1}}dy \right.$$
$$\left. +(\gamma+2)\int_{-1}^{1} g(y)\frac{(1+y)^{\gamma+1}+(1-y)^{\gamma+1}}{\pi(1-y^2)^{\gamma+1}}dy\right\} + G_2(\gamma), \qquad (33)$$

is true, where $G_2(\gamma)$ is a function meromorphic in the half-plane $\mathfrak{Re}\,\gamma > -2$ having simple poles at $\gamma = 2k-1$, $k = 1, 2, \ldots$ which satisfies estimates (32) outside the circles $|\gamma - 2k + 1/ \le 1$.

Proof. Taking into account (5) and (22), we obtain for $\mathfrak{Re}\,\gamma \in (-3, 2)$:

$$G_1(\gamma) = \frac{2\pi}{\cos\frac{\pi\gamma}{2}}\left\{(\gamma+1)\int_{-1}^{1} f'(y)S(\gamma+1, y)dy - (\gamma+2)\int_{-1}^{1} g(y)C(\gamma+1, y)dy\right\}. \qquad (34)$$

The Hölder inequality

$$\|l(y)(1-|y|)^{-\mu}\|_{L_1} \le c_\mu \|l(y)\|_{L_p}, \ \ \forall\mu < 1/p'.$$

and Condition (3) garantee that the function $G_1(\gamma)$ has no pole at $\gamma = -1$. Hence, using Proposition 1, we conclude that the function $G_1(\gamma)$ is analytic in the strip $\mathfrak{Re}(-3, -1-1/p)$ and satisfies estimates (32) with the constant c dependent on $N_p^1(f, g)$. Now relation (33) follows from (34), (31) and (3). The Lemma is thus proved.

From (32) and [26] we obtain for $p > 2$ the following specification of estimate (27) applied to the function $F_1(\lambda)$:

$$|F_1(\lambda)| \le c_{\epsilon,\delta} N_p^1(f, g)|\lambda|^3(1+|\lambda|)^{-2+1/p+\delta}, \ \lambda \in \Sigma_\epsilon, \ \forall\delta > 0. \qquad (35)$$

Theorem 2. The Mellin transform

$$M(\gamma) = M([X](\gamma) = \int_0^\infty X(s)s^{\gamma-1}ds$$

of the solution of Equation (18) is analytic in the strip $\mathfrak{Re}\,\gamma \in (-4, -1-1/p)$ and for any $\mu > 0$ the following estimate is true:

$$|M(\gamma)| \le c_\mu N_p^1(f, g)e^{-\delta_1|\mathfrak{Im}\,\gamma|}, \quad , \mathfrak{Re}\,\gamma \in (-4+\mu, -1-1/p-\mu), \qquad (36)$$

with a constant $\delta_1 > 0$. For $\mathfrak{Re}\,\gamma > \sigma_1$ the following relation is valid

$$M(\gamma) = M_0(\gamma) + M_1(\gamma), \tag{37}$$

where

$$M_0(\gamma) = \frac{2\Gamma(\gamma+2)\cos\frac{\pi\gamma}{2}}{D_0(\gamma)}\left\{(\gamma+1)\sin\frac{\pi\gamma}{2}\int_{-1}^{1} f(y)\frac{(1+y)^{\gamma+1}-(1-y)^{\gamma+1}}{\pi(1-y^2)^{\gamma+1}}\mathrm{d}y\right.$$
$$\left.+(\gamma+2)\cos\frac{\pi\gamma}{2}\int_{-1}^{1} g(y)\frac{(1+y^{\gamma+1}+(1-y)^{\gamma+1}}{\pi(1-y^2)^{\gamma+1}}\mathrm{d}y\right\} + G_2(y),$$

and $M_1(\gamma)$ is a function meromorphic in the half-plane $\mathfrak{Re}\,\gamma > \sigma_1$ having poles at the roots $\gamma_0 = 0$, $\gamma_k, k = 1, 2, \ldots$ of the function $D_0(\gamma)$ located in $\mathfrak{Re}\,\gamma \geq 0$, which admits the estimate

$$|M_1(\gamma)| \leq c_\mu N_p^1(f, g)\mathrm{e}^{-\delta_1|\mathfrak{Im}\,\gamma|}, \quad \mathfrak{Re}\,\gamma \in (\sigma_1 + 1/\mu, \mu).$$

Proof. Using [26, Section 1.29], we obtain from Lemma 1 and Corollary 1 the statement about analyticity $M(\gamma)$ in the strip $\mathfrak{Re}\,\gamma \in (-4, -1 - 1/p_0)$, together with estimates (36). Then from (16)–(18) we obtain that the function $X(s)$ satisfies the equation (see (28)):

$$X(s) - \int_0^\infty Q_0(s,t)X(t)\mathrm{d}t = F_1(s) + F_2(s), \quad s > 0$$

with

$$F_2(s) = \left(1 - \frac{\sinh^2 s}{\Delta(s)}\right)\cdot\left(\int_0^\infty Q_0(s,t)X(t)\mathrm{d}t - F_1(s)\right).$$

From (21), (21), (24) and [26] we conclude that the Mellin transform $M[F_2](\gamma)$ is an analytic function in $\mathfrak{Re}\,\gamma > -3$ satisfying the estimates

$$|M[F_2](\gamma)| \leq c_\mu N_p^1(f, g)\mathrm{e}^{-\delta_1|\mathfrak{Im}\,\gamma|}, \ \mathfrak{Re}\,\gamma \in (-2, \mu), \ \forall\mu > 0.$$

Using [24], Lemma 2 and formulae (29), (30) we obtain for $M(\gamma)$ the representation

$$M(\gamma)\frac{D_0(\gamma)}{\cos^2\pi\gamma/2} = \frac{\pi^\gamma(\gamma+1)}{2\pi\mathrm{i}\,\cos\pi\gamma/2}\int_{\sigma_1-\mathrm{i}\infty}^{\sigma_1+\mathrm{i}\infty}\frac{\pi^{-\xi}(\xi+1)}{\cos\pi\xi/2}\zeta(\xi-\gamma+1)M(\xi)\mathrm{d}\xi +$$
$$+G_1(\gamma) + M[F_2](\gamma),$$

where $\mathfrak{Re}\,\gamma \in (\sigma_1, -1 - 1/p)$ and $\zeta(z)$ is the Riemann ζ-function. Analysing the last expression together with (33), we complete the proof.

Corollary 2. For the Solution $X(s)$ of Equation (18) and for the corresponding sequence X_n the representations

$$X(s) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} M(\gamma)s^{-\gamma}\mathrm{d}\gamma, \quad s > 0, \quad \sigma \in (-4, -1 - 1/p),$$

$$X_n = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} M(\gamma)\frac{(\gamma+1)}{\cos\frac{\pi\gamma}{2}}\alpha_n^{-\gamma}\mathrm{d}\gamma, \ n = 1, 2, \ldots, \sigma \in (-3, -1 - 1/p). \tag{38}$$

are valid, having the asymptotics

$$X(s) = \mathrm{O}(s^{\sigma}), \quad s \to \infty, \quad X_n = \mathrm{O}(n^{\sigma}), \quad n \to \infty, \quad \forall \sigma > 1 + 1/p. \tag{39}$$

Let $X(s)$ be the solution of Equation (18) and let X_n be the corresponding sequence from (14). Then, in particular, the conditions (10) are fulfilled and, therefore the function u from (6–8) is indefinitely differentiable for $(x, y) \in S$ and satisfies there the biharmonic equation. Let us prove that u also satisfies the boundary conditions from (1) at the lateral boundaries of the half-strip and at infinity.

Using the equation

$$U(\lambda, y) = \lambda \cosh \lambda(1 - |y|) + \lambda(1 - |y|) \sinh \lambda \sinh \lambda|y| + \sinh \lambda \cosh \lambda y,$$

we obtain the estimates

$$\left| \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} U(\lambda, y) \right| \leq c_n (1 + |\lambda|)^n (1 + |\lambda|(1 - |y|)\mathrm{e}^{|\mathfrak{Re}\,\lambda|(1-|y|)}, \quad |y| \leq 1. \tag{40}$$

Due to (27) and (24) we obtain from (26) the relation

$$T(\lambda) = \mathrm{o}(|\lambda|^{-1}), \quad \lambda \to \infty, \quad \lambda \in \Sigma_{\epsilon_0}. \tag{41}$$

Then using (26), (40), (41) and Jordan's lemma, we obtain for the function u_1 from (7) the representation

$$u_1(x, y) = \frac{1}{\pi} \left\{ \int_{\mathrm{i}\delta - \mathrm{e}^{-\mathrm{i}\theta}\infty}^{\mathrm{i}\delta} + \int_{\mathrm{i}\delta}^{\mathrm{i}\delta + \mathrm{e}^{-\mathrm{i}\theta}\infty} \right\} \frac{T(\lambda) U(\lambda, y)}{\Delta(y)} \mathrm{e}^{\mathrm{i}\lambda x} \mathrm{d}\lambda, \tag{42}$$

for $x > 0$, $|y| < 1$ and some $\delta \in (0, \min(\alpha_1, \mathfrak{Im}\,\lambda_k))$, $\theta\, n(0, \pi/2)$ In turn, it follows from (42), (6)–(8) that the function $u(x, y)$ is indefinitely differentiable for $x > 0$, $|y| \leq 1$ and due to Corollary 1 and (24) it fulfills the estimate

$$|u(x, y)| \leq c N_p^1(f, g) \mathrm{e}^{-\delta x}, \quad x \geq 1, \ \ |y| \leq 1. \tag{43}$$

Then from (42), the equality $U(\lambda, 1) = \Delta(\lambda)$, $U'_u(\lambda, 1) = 0$, $\lambda \in \mathrm{C}$ and (11), we obtain

$$u(x, \pm 1) = u'_y(x, \pm 1) = 0, \quad x > 0,$$

i.e., the boundary conditions from (1) at surfaces $y = \pm 1$ are satisfied.

It should be noticed that estimate (43) really continues to be valid on replacing the constant $\delta \in (0, \min(\alpha_1, \mathfrak{Im}\,\lambda_k))$ by the exact constant $\delta = \min(\mathfrak{Im}\,\lambda_k)$ (see [11]).

4. Satisfaction of the non-zero boundary conditions

Let us consider the validity of the boundary conditions (1) on the edge $x = 0$ of the half-strip. Relations (37)–(39) play an important role in obtaining the following main result of this subsection.

Theorem 3. The following is true for the function $u(x, y)$ from (6)–(8):

$$\|u(x, y) - f(y)\|_{W_p^1} + \|u_x(x, y) - g(y)\|_{L_p} \to 0, \quad x \to 0. \tag{44}$$

To prove (44) let us make the same preliminary constructions as connected with the transformation of expressions (6)–(8). Consider the function

$$v(x,y) = -f_0 + \frac{1}{2\pi}\int_0^\infty X(s)\frac{U(x,y)}{s^3 \sinh^2 s}\cos(xs)\mathrm{d}s - \\ -\frac{1}{2}\sum_{n=1}^\infty \frac{(-1)^n}{\alpha_n^3}X_n(+\alpha_n x)\mathrm{e}^{-\alpha_n x}\cos(\alpha_n x), \;\; x > 0, \;\; |y| < 1. \tag{45}$$

Using (6)–(8) we obtain

$$u(x,y) = v(x,y) + f_0 + \sum_{n=1}^\infty \{f_n(1+\alpha_n x) + g_n x\}\mathrm{e}^{\alpha_n x}\cos(\alpha_n y). \tag{46}$$

For $x > 0$, $|y| < 1$ we have the integral representation

$$C(\gamma; x, y) \equiv \sum_{n=1}^\infty (-1)^n \alpha_n^\gamma \mathrm{e}^{-\alpha_n x} = \frac{1}{\pi}\int_0^\infty \frac{s^\gamma \cosh sy}{\sinh s}\sin(xs - \pi\gamma/2)\mathrm{d}s, \tag{47}$$

with $\mathfrak{Re}\,\gamma > 0$. Denote by

$$R(\gamma, t) = \gamma t \sin t + (\gamma + 1)\tan \pi\gamma/2(t \cos t - \sin t).$$

Lemma 3. For $\sigma \in (-2. -1 - 1/p)$ the function $v(x,y)$ can be presented in the form:

$$v(x,y) = \frac{1}{4\pi^2 i}\int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} M(\gamma)\left\{\int_0^\infty \frac{s^{-\gamma-3}\cosh sy}{\sinh s}R(\gamma, xs)\mathrm{d}s\right\}\mathrm{d}\gamma. \tag{48}$$

Proof. To simplify the formulae we give the proof for the case of $f_0 = 0$. Then (15) implies $M(-3) = 0$ and the sequence X_n satisfies relation (38) for all $\sigma \in (-4, -1 - 1/p)$. In this case substituting (38) with some $\sigma \in (-4, -3)$ in (45) and using the Fubini theorem, we obtain for $x > 0$, $|y| < 1$:

$$v(x,y) = \frac{1}{4\pi^2 i}\int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} M(\gamma)\left\{\int_0^\infty s^{-\gamma-3}\frac{U(x,y)}{\sinh^2 s}\cos(xs)\mathrm{d}s - \right. \\ \left. -\frac{\pi(\gamma+1)}{\cos \pi\gamma/2}(C(-\gamma-3; x, y) + xC(-\gamma-2; x, y))\right\}\mathrm{d}\gamma. \tag{49}$$

Then, on integrating by parts, we get

$$\int_0^\infty s^{-(\gamma+3)}\frac{U(s,y)}{\sinh^2 s}\cos(xs)\mathrm{d}s = -\int_0^\infty s^{-(\gamma+2)}\frac{\cosh sy}{\sinh s}(xs \sin(xs) + (\gamma+1)\cos(xs))\mathrm{d}s.$$

Using the last expression and expression (47), we obtain from (49) that (48) is valid for $\sigma \in (-4, -3)$. Since the function $M(\gamma)R(\gamma, t)$ is analytic in the strip $\mathfrak{Re}\gamma \in (-4, -1, -1/p)$ and $R(\gamma, t)$ has a second-order zero at $t = 0$, the Cauchy theorem (with account of Theorem 2) implies (48) for all $\sigma \in (-2, -1, -1/p)$. The Lemma is proved.

Using the relation

$$\frac{1}{\sinh s} = 2\mathrm{e}^{-s} + \frac{\mathrm{e}^{-2s}}{\sinh s},$$

and [22] we obtain for $\mathfrak{Re}\gamma \in (-2, -1)$, $y \in (0, 1)$:

$$\int_0^\infty \frac{s^{-\gamma-3}\cosh sy}{\sinh s}R(\gamma, xs)\mathrm{d}s = \gamma\Gamma(-\gamma-1)r_1^{\gamma+2}(x,y)P_0(\gamma; x(1-y)^{-1}) + P_1(\gamma; x, y), \tag{50}$$

where

$$r_1(x,y) = (x^2 + (1-y)^2)^{1/2},$$

$$P_0(\gamma;t) = -\frac{t}{\sqrt{1+t^2}} \sin((\gamma+1)\arctan t) +$$
$$+\gamma^{-1}(\gamma+1)\tan \pi\gamma/2 \left\{ \frac{t\ \cos((\gamma+1)\arctan t)}{\sqrt{1+t^2}} - \frac{\sin((\gamma+2)\arctan t)}{\gamma+2} \right\}, \quad t>0,$$

and $P_1(\gamma;x,y)$ is an infinitely differentiable with respect to $x \geq 0$, $y \in [0,1]$ function satisfying the estimates

$$\|P_1(\gamma;x,y)\|_{C^k[0,1]} + \|\partial/\partial x P_1(\gamma;x,y)\|_{C^k[0,1]} \leq c_{k,\mu} x|\gamma|,$$

$$\mathfrak{Re}\gamma \in [-2,-1-\mu), \quad \forall\mu > 0.$$

Lemma 4. The following relation is true

$$\|v(x,y)\|_{W_p^1} + \|v_x(x,y)\|_{L_p} \to 0, \quad x \to 0. \tag{51}$$

Proof. The function $v(x,y)$ is even with respect to y and, therefore, due to (48), (50), (37) it is sufficient to prove (51) for the functions

$$v_j(x,y) = \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} M_j(\gamma)\gamma\Gamma(-\gamma-1) r_1^{\gamma+2}(x,y) P_0(\gamma; x(1-y)^{-1})\mathrm{d}\gamma, \ \sigma \in (-2,-1-1/p).$$

Using the properties of the functions $M_0(\gamma)$ and $M_1(\gamma)$ (see Theorem 2) and the expressions

$$P_0(-1;t) = 0, \quad P_0(0;t) = \frac{t^2}{1+t^2}, \quad \lim_{\gamma\to 1}(\gamma-1)P_0(\gamma,t) = -\frac{4}{\pi}\left(\frac{t}{\sqrt{1+t^2}}\right)^3,$$

we obtain

$$v_0(x,y) = -\pi \int_{\sigma-\mathrm{i}\infty}^{\sigma-\mathrm{i}\infty} \frac{\gamma}{D_0(\gamma)} \left\{ (\gamma+1)\int_{-1}^{1} f'(t) \frac{(1+t)^{\gamma+2} - (1-t)^{\gamma+2}}{(1-t^2)^{\gamma+2}} \mathrm{d}t + \right. \tag{52}$$
$$\left. +(\gamma+2)\cot \pi\gamma/2 \int_{-1}^{1} g'(t) \frac{(1+t)^{\gamma+2} + (1-t)^{\gamma+2}}{(1-t^2)^{\gamma+2}} \mathrm{d}t \right\} r_1^{\gamma+2}(x,y) P_0(\gamma; x(1-y))\mathrm{d}\gamma,$$

$$v_1(x,y) = c_0x^2 + c_1x^3 + \int_{\sigma_2-\mathrm{i}\infty}^{\sigma_2-\mathrm{i}\infty} M_1(\gamma)\gamma\Gamma(-\gamma-1) r_1^{\gamma+2}(x,y) P_0(\gamma; x(1-y))\mathrm{d}\gamma, \ \sigma_2 \in (1,\sigma_0)$$

(see (25)). Then, on using the estimates,

$$\left|\frac{\partial^k}{\partial t^k} P_0(\gamma;t)\right| \leq c \frac{t^{2-k}}{(1+t^2)^{2+k}} (1+|\gamma|^2) e^{\pi/2|\mathfrak{Im}\gamma|}, \ \mathfrak{Re}\gamma \leq 2, \ |\gamma-1| \geq 1, \ k = 0,1, \tag{53}$$

and the estimates for $M_1(\gamma)$ from Theorem 2 and the estimate for the gamma function [25]

$$|\Gamma(-\gamma-1)| \leq c e^{-\pi/2|\mathfrak{Im}\gamma|}, \quad \mathfrak{Re}\,\gamma = \sigma_2,$$

we obtain from (52)

$$\|v_1(x,y)\|_{C^3[0,1]} + \|\partial/\partial x v_1(x,y)\|_{C^2[0,1]} \to 0, \quad x \to 0,$$

which, in particular, is the statement (51) for the function $v_1(x, y)$.

Then, using (52) and (53), we may easily show that statement (51) for the function $v_0(x, y)$ is equivalent to

$$\sum_{j=0}^{2} \|K_j(x)l\|_{L_p[0,1]} \to 0, \quad x \to 0, \quad \forall\, l(y) \in L_p[0,1], \tag{54}$$

where the integral operators

$$(K_j(x)l)(y) = \int_0^1 K_j(x; y, t)l(t)\mathrm{d}t$$

have the kernels

$$K_0(x; y, t) = \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathcal{K}(\gamma)(x^2+y^2)^{\gamma/2+1} P_0(\gamma; xy^{-1})t^{-\gamma-2}\mathrm{d}\gamma, \tag{55}$$

$$K_1(x; y, t) = \frac{\partial}{\partial y} K_0(x; y, t), \quad K_2(x; y, t) = \frac{\partial}{\partial x} K_0(x; y, t),$$

and $\mathcal{K}(\gamma)$ is analytic in the strip $\mathfrak{Re}\gamma \in (-2, -1)$, satisfying the estimates

$$|\mathcal{K}(\gamma)| \le c_\mu |\gamma|^2 \mathrm{e}^{-\pi|\mathfrak{Im}\gamma|}, \quad \mathfrak{Re}\gamma \in (-2+\mu, -1-\mu), \ \forall\, \mu > 0.$$

If the function $l(y) \in C_0^1[0,1]$, then integrating by parts by t in the expressions for $(K_j(x)l)(y)$ and taking into account (53), (55), we obtain for $x \in (0,1]$, $\sigma \in (-1-1/p, -1)$:

$$\sum_{j=0}^{2} \|K_j(x)l\|_{L_p[0,1]} \le c\|l'(t)\|_{L_p[0,1]} \cdot x \left(\int_0^1 (x+y)^{\sigma p}\mathrm{d}y\right)^{1/p} \le cx^{\sigma+1+1/p} \to 0,$$

with $x \to 0$. The set $C_0^1[0,1]$ is dense in the space $L_p[0,1]$. Therefore, due to the Banach-Steinhaus theorem, to complete the proof of the lemma, it remains to prove uniformity with respect to $x \in (0,1]$ and boundedness in $L_p[0,1]$ of the set of integral operators $K_j(x)$, $j = 0, 1, 2$. Using (53), (55) we obtain the following estimates for $x \in (0,1]$, $0 < y, t < 1$ for the kernels

$$\sum_{j=0}^{2} |K_j(x; y, t)| \le c_\sigma \frac{y^{\sigma+1}}{t^{\sigma+2}}, \quad \sigma \in (-2, -1). \tag{56}$$

Then choosing for $0 < t < y$ in (56) the value of the parameter σ from the interval $(-2, -1-1/p)$, and for $0 < y < t$ the value of σ from $(-1-1/p, -1)$, we obtain due to [27] the following statement

$$\sum_{j=0}^{2} \|K_j(x)\|_{L_p[0,1]} \le c, \quad x \in (0,1].$$

The lemma is proved.

Proof of Theorem 3. Due to (46), Lemma 4 and (5), to prove the theorem it is sufficient to deduce the relations

$$\|f_0 + \sum_{n=1}^{\infty} f_n e^{-\alpha_n} \cos(\alpha_n y) - f(y)\|_{W_p^1} + \|\sum_{n=1}^{\infty} g_n e^{-\alpha_n} \cos(\alpha_n y) - g(y)\|_{L_p} \to 0,$$

$$x\|\sum_{n=1}^{\infty} f_n^{(1)} \alpha_n^j e^{-\alpha_n} e^{\mathrm{i}\alpha_n y}\|_{L_p} + x\|\sum_{n=1}^{\infty} g_n \alpha_n^j e^{-\alpha_n} e^{\mathrm{i}\alpha_n y}\|_{L_p} \to 0,$$

with $x \to 0$ and $j = 0, 1$. These relations follow from (2)–(3), from the properties of the Poisson kernel

$$P_r(\mathrm{e}^{\mathrm{i}t}) = \frac{1 - r^2}{1 - 2r\ \cos\ t + r^2}, \quad r \in [0, 1), \quad |t| < \pi$$

and from the boundedness of the Hilbert transform in L_p, $1 < p < \infty$ [27]. The Theorem is proved.

Analysing the proof of Lemma 4, we should make the following remark. If we use local polar coordinates

$$x = r_1\ \sin\theta, \quad y = 1 - r_1\ \sin\theta, \quad r_1 \in (0, 1), \quad \theta \in (0, \pi/2),$$

in the neighbourhood of the corner point $x = 0$, $y = 1$ of the half-strip, we have

$$\begin{aligned} &P_0(\gamma; x(1-y)) = \psi(\gamma, \theta) \\ &= -\sin\theta\ \sin(\gamma+1)\theta + \frac{\gamma+1}{\gamma}\tan\frac{\pi\gamma}{2}\left[\sin\theta\ \cos(\gamma+1)\theta - \frac{\sin(\gamma+2)\theta}{\gamma+2}\right]. \end{aligned} \tag{57}$$

It follows from (48), (50), (52) and Theorem 2 that in the neighbourhood of the point $x = 0$, $y = 1$ the function $v(x, y)$ admits the representation

$$v(x, y) = \sum_{k=1}^{n-1} b_k r_1^{\gamma_k+2} \psi(\gamma_k, \theta) + v_0(x, y) + w(x, y) + \mathrm{O}(r_1^{\mathfrak{Re}\gamma_N+2}), \tag{58}$$

with $r_1 \to 0$, $(r_1, \theta) \in V$, where $V = \{r_1 \in (0, 1), \theta \in (0, \pi/2)\}$, $w \in C^{\infty}(\overline{V})$ and $v_0 \in C^{\infty}(V)$ is the function defined in the proof of Lemma 4. The function $r_1^{\gamma_k+2}\psi(\gamma_k, \theta)$, where γ_k are the roots of $D_0(\gamma)$ with $\mathfrak{Re}\,\gamma_k > 0$, is the solution of the homogeneous Dirichlet problem in the wedge $r_1 > 0$, $\theta \in (0, \pi/2)$. Hence, by (46), (52), (58), the full description of the local behaviour of the solution of the boundary-value problem (1) is given in the neighbourhood of the corner $x = 0$, $y = 1$.

5. Remarks

It should be noted that condition (3) does not restrict the generality of the analysis, because it can be removed with the help of the partial solution of the biharmonic equation of the form $u_0(x, y) = U(\lambda_k, y)\mathrm{e}^{\mathrm{i}\lambda_k x}$, which satisfies homogeneous Dirichlet conditions at the sides $y = \pm 1$ of the half-strip. Generalizing the above methods, we obtain the following theorem.

Theorem 4. Let even functions f, g satisfy the conditions

$$f \in W_p^k, \quad g \in W_p^{k-1}, \quad 1 < p < \infty,$$

for some $k = 1, 2, 3$ and let

$$f(1) = 0, \quad k = 1; \quad f(1) = f'(1) = g(1) = 0, \quad k = 2;$$

$$f(1) = f'(1) = g(1) = g'(1) = 0, \qquad k = 3.$$

Then there exists an infinitely differentiable solution u of boundary-value problem (1) for $x < 0$, $|y| \le 1$, estimated by (43) and satisfying the relation

$$\|u(x, y) - f(y)\|_{W_p^k} + \|u_x(x, y) - g(y)\|_{W_p^{k-1}} \to 0, \qquad x \to 0.$$

This solution is unique and there exists a constant $c > 0$, independent of f and g, such that the estimate

$$\|u(x, y)\|_{W_p^k} + \|u_x(x, y)\|_{W_p^{k-1}} \le cN_p^k(f, g), \qquad \forall x > 0 \tag{59}$$

is valid. If g satisfies condition (3), then the solution u is given by (6)–(8), where the function $X(s)$ and the sequence X_n are defined according to Corollary 1 and (14).

We notice that we used the method of [28] to prove uniqueness as well as the estimate (59), which was originally established for the solutions of the form (6)–(8).

The case should be noticed where the solution of the integral equation (18) can be given in explicit form. Namely, let the real function $F(s)$ be continuous for $s \ge 0$ and let it admit the estimate

$$|F(s)| \le cs^{-1}, \qquad s \ge 1. \tag{60}$$

Then, from the regularity properties of the kernel $Q(s, t)$, using the general results of L.V. Kantorovich on the solvability of functional equations in semi-ordered spaces [29] applied to Equation (18) in the real K-space $L_\infty(R_+)$, we obtain that Equation (18) possesses a continuous solution $X(s)$ bounded for $s \ge 0$ represented by the Neumann series

$$X(s) = F(s) + \sum_{n=1}^{\infty} \int_0^\infty Q_n(s, t)F(t)\mathrm{d}t, \qquad s \ge 0 \tag{61}$$

where $(Q_n(s, t)$ is the n-th iteration of the kernel $Q(s, t))$, which is convergent in the space $C[0, d]$ for any finite $d > 0$. In the case where the function $F(s)$ is of the form (16) and (60) is valid, this solution is unique due to Corollary 1.

It is possible to show that, if the functions f and g satisfy the conditions of Theorem 4 with $k = 1$ and $f^{(3)}(1) = 0$, then the function $F(s)$ of the form (16), satisfies condition (60). Condition (60) is satisfied also if f, g satisfy the conditions of Theorem 4 with $k = 1$ under the additional conditions $f(y) = g(y) = 0$, $|y| \in (y_0, 1]$ for some $y_0 \in (0, 1)$. In both cases we obtain from (38) the following specification of the asymptotic formulae (39), namely, for any $\sigma < \sigma_0 \in (1, 2)$:

$$X(s) = a + \mathrm{O}(s^{-\sigma}), \qquad s \to \infty, X_n = a + \mathrm{O}(n^{-\sigma}), \qquad n \to \infty, \tag{62}$$

where a is a constant, and the value of σ_0 is defined by (30). Statement (62) makes it possible to use the method of improved reduction to solve numerically the system (13), (14), (see [18], [30, Chapter 1]).

We notice that in the same way, using the method of superposition, it is possible to investigate the antisymmetric boundary-value problem (1), *i.e.*, problem (1) with the odd functions f, g. In this case we look for a solution u of the form

$$u(x, y) = \frac{1}{2\pi} \int_0^\infty X(s) \frac{(\cosh s \ \sinh sy - y \ \sinh s \ \cosh sy)}{s^2 \ \sinh^2 s} \sin(sx) \mathrm{d}s +$$

$$+ \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n X_n}{2\alpha_n^2} x(1 + \alpha_n + x) f_n + g_n^x \right\} \mathrm{e}^{-\alpha_n x} \sin(\alpha_n y),$$

where

$$f(y) = \sum_{n=1}^{\infty} f_n \sin(\alpha_n y), \quad g(y) = \sum_{n=1}^{\infty} g_n \sin(\alpha_n y).$$

At the same time the function $X(s)$ must satisfy integral equation (18) with the kernel

$$Q(s, t) = \frac{16 s^3 \sinh^2 s}{\pi(\sinh s \ \cosh s - s)} \sum_{n=1}^{\infty} \frac{\alpha_n^3}{(s^2 + \alpha_n^2)^2 (t^2 + \alpha_n^2)^2}$$

and the right-hand side

$$F(s) = \frac{4s^3 \ \sinh^2 s}{(\sinh s \ \cosh s - s)} \sum_{n=1}^{\infty} \frac{(-1)^n \alpha_n \{(3\alpha_n^2 + s^2) f_n + 2\alpha_n g_n\}}{(s^2 + \alpha_n^2)^2}.$$

Expressions (14) for the sequence X_n remain true. The nonnegative kernel $Q(s, t)$ satisfies the regularity condition

$$1 - \int_0^\infty Q(s, t) \mathrm{d}t = \frac{2(\sinh^2 - s^2)}{s(\sinh s \ \cosh s - s)} > 0, \quad s \geq 0,$$

and its main part is of the form (28).

6. Conclusion

In this paper we have given a mathematical justification for the applicability of the method of the superposition to the Dirichlet problem for the biharmonic equation in the semi-infinite strip. The proposed way of consideration of the integral equation of the method of superposition can be extended to non-smooth boundary functions at the short end. This might be useful for application in various problems of the theory of elasticity, bending of thin plates and Stokes flow.

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